Strong Admissibility Revisited

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Abstract. In the current paper, we re-examine the concept of *strong admissibility*, as was originally introduced by Baroni and Giacomin. We examine the formal properties of strong admissibility, both in its extension-based and in its labelling-based form. Moreover, we show that strong admissibility plays a vital role in discussion-based proof procedures for grounded semantics. In particular it allows one to compare the performance of alternative dialectical proof procedures for grounded semantics, and obtain some remarkable differences between the Standard Grounded Game and the Grounded Persuasion Game.

Keywords. strong admissibility, grounded semantics, argument games

1. Introduction

Admissibility is generally seen as one of the cornerstones of abstract argumentation theory [15], as it is the basis of various argumentation semantics [1]. Not only does admissibility appeal to common intuitions [3], it is also one of the key requirements for obtaining a consistent outcome of instantiated argumentation formalisms [8,17,19].

Slightly less well-known is the principle of *strong admissibility*, which was originally introduced in [2]. The original aim of strong admissibility was to characterise the unique properties of the grounded extension. It turns out, however, that the concept is also useful for comparing the characteristics of the different dialectical proof procedures that have been stated in the literature. In particular, the Standard Grounded Game [16,20] and the Grounded Persuasion Game [13,14] prove membership of the grounded extension essentially by constructing a strongly admissible set around the argument in question. However, as we will see, the Grounded Persuasion Game is able to do so in a more efficient way, requiring a number of steps that is *linearly* related to the `in`/`out`-size[1] of the strongly admissible set, whereas the Standard Grounded Game can require a number of steps that is *exponentially* related to the `in`/`out`-size of the strongly admissible set.

The remaining part of the current paper is structured as follows.[2] First, in Section 2 we briefly summarise some of the key concepts of abstract argumentation theory, both in its extension and in its labelling based form. In Section 3, we then discuss the extension based version of strong admissibility and examine its formal properties. In Section 4 we introduce the labelling based version of strong admissibility and show how it relates to its extension based version. In Section 5 we then re-examine the Standard Grounded Game,

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[1] By the `in`/`out`-size of a set of arguments, we mean the number of arguments in the set itself plus the number of arguments attacking the set.

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and the Grounded Persuasion Game, and show that strong admissibility plays a vital role in describing the relative efficiency of these games. In Section 6 we then round off with a discussion of the obtained results and open issues. Although some of the proofs had to be omitted due to lack of space, these can be found in a seperate technical report [7].

2. Formal Preliminaries

In the current section, we briefly restate some of the key concepts of abstract argumentation theory, in both its extension based and labelling based form.

Definition 1 ([15])**.** *An* argumentation framework *is a pair* (Ar, att) *where* Ar *is a finite set of entities, called arguments, whose internal structure can be left unspecified, and* att *a binary relation on* Ar*. We say that* A attacks B *iff* $(A, B) \in att$*.*

Definition 2. *Let* (Ar, att) *be an argumentation framework,* $A \in Ar$ *and* $\mathcal{A}rgs \subseteq Ar$*. We define* A^+ *as* $\{B \in Ar \mid A \text{ attacks } B\}$*,* A^- *as* $\{B \in Ar \mid B \text{ attacks } A\}$*,* $\mathcal{A}rgs^+$ *as* $\cup\{A^+ \mid A \in \mathcal{A}rgs\}$*, and* $\mathcal{A}rgs^-$ *as* $\cup\{A^- \mid A \in \mathcal{A}rgs\}$*.* $\mathcal{A}rgs$ *is said to be* conflict-free *iff* $\mathcal{A}rgs \cap \mathcal{A}rgs^+ = \emptyset$*.* $\mathcal{A}rgs$ *is said to* defend A *iff* $A^- \subseteq \mathcal{A}rgs^+$*. The characteristic function* $F : 2^{Ar} \to 2^{Ar}$ *is defined as* $F(\mathcal{A}rgs) = \{A \mid \mathcal{A}rgs \text{ defends } A\}$*.*

Definition 3. *Let* (Ar, att) *be an argumentation framework.* $\mathcal{A}rgs \subseteq Ar$ *is said to be:*
  * *an admissible set iff* $\mathcal{A}rgs$ *is conflict-free and* $\mathcal{A}rgs \subseteq F(\mathcal{A}rgs)$
  * *a complete extension iff* $\mathcal{A}rgs$ *is conflict-free and* $\mathcal{A}rgs = F(\mathcal{A}rgs)$
  * *a grounded extension iff* $\mathcal{A}rgs$ *is the smallest (w.r.t.* $\subseteq$*) complete extension*
  * *a preferred extension iff* $\mathcal{A}rgs$ *is a maximal (w.r.t.* $\subseteq$*) complete extension*

If $\mathcal{A}rgs$ is a conflict-free set, then its *down-admissible* set (written as $\mathcal{A}rgs\!\downarrow$) is defined as the (unique) biggest admissible subset of $\mathcal{A}rgs$.[3]
The above definitions essentially follow the extension based approach of [15]. It is also possible to define the key argumentation concepts in terms of argument labellings [5,10].

Definition 4. *Let* (Ar, att) *be an argumentation framework. An* argument labelling *is a partial function* $\mathcal{L}ab : Ar \to \{\texttt{in}, \texttt{out}, \texttt{undec}\}$*. An argument labelling is called an* admissible labelling *iff* $\mathcal{L}ab$ *is a total function and for each* $A \in Ar$ *it holds that:*
  * *if* $\mathcal{L}ab(A) = \texttt{in}$ *then for each* B *that attacks* A *it holds that* $\mathcal{L}ab(B) = \texttt{out}$
  * *if* $\mathcal{L}ab(A) = \texttt{out}$ *then there exists a* B *that attacks* A *such that* $\mathcal{L}ab(B) = \texttt{in}$

$\mathcal{L}ab$ *is called a* complete labelling *iff it is an admissible labelling and for each* $A \in Ar$ *it also holds that:*
  * *if* $\mathcal{L}ab(A) = \texttt{undec}$ *then not for each* B *that attacks* A *it holds that* $\mathcal{L}ab(B) = \texttt{out}$*, and there exists no* B *that attacks* A *such that* $\mathcal{L}ab(B) = \texttt{in}$

As a labelling is essentially a function, we sometimes write it as a set of pairs. Also, if $\mathcal{L}ab$ is a labelling, we write $\texttt{in}(\mathcal{L}ab)$ for $\{A \in Ar \mid \mathcal{L}ab(A) = \texttt{in}\}$, $\texttt{out}(\mathcal{L}ab)$ for $\{A \in Ar \mid \mathcal{L}ab(A) = \texttt{out}\}$ and $\texttt{undec}(\mathcal{L}ab)$ for $\{A \in Ar \mid \mathcal{L}ab(A) = \texttt{undec}\}$. As a labelling is also a partition of the arguments into sets of $\texttt{in}$-labelled arguments,

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[3]The well-definedness of the down-admissible set follows from [11], where this concept is defined in its labellings form, together with the equivalence between extensions and labellings [10].

out-labelled arguments and undec-labelled arguments, we sometimes write it as a triplet $(\mathtt{in}(\mathcal{L}ab), \mathtt{out}(\mathcal{L}ab), \mathtt{undec}(\mathcal{L}ab))$.

Definition 5 ([11]). *Let $\mathcal{L}ab$ and $\mathcal{L}ab'$ be argument labellings of argumentation framework (Ar, att). We say that $\mathcal{L}ab \sqsubseteq \mathcal{L}ab'$ iff $\mathtt{in}(\mathcal{L}ab) \subseteq \mathtt{in}(\mathcal{L}ab')$ and $\mathtt{out}(\mathcal{L}ab) \subseteq \mathtt{out}(\mathcal{L}ab')$.*

We say that $\mathcal{L}ab_1$ is a sublabelling of $\mathcal{L}ab_2$ (or alternatively, that $\mathcal{L}ab_2$ is a superlabelling of $\mathcal{L}ab_2$) iff $\mathcal{L}ab_1 \sqsubseteq \mathcal{L}ab_2$. If $\mathcal{L}ab$ is a total labelling (*i.e.* a total function), then its *down-admissible* labelling [11] (written as $\mathcal{L}ab{\downarrow}$) is defined as the (unique) biggest (w.r.t. $\sqsubseteq$) admissible sublabelling of $\mathcal{L}ab$.

Definition 6. *Let $\mathcal{L}ab$ be a complete labelling of argumentation framework (Ar, att). $\mathcal{L}ab$ is said to be*
- *a grounded labelling iff $\mathcal{L}ab$ is the (unique) smallest (w.r.t. $\sqsubseteq$) complete labelling*
- *a preferred labelling iff $\mathcal{L}ab$ is a maximal (w.r.t. $\sqsubseteq$) complete labelling*

Given an argumentation framework (Ar, att) we define two functions Args2Lab and Lab2Args (to translate a conflict-free set of arguments to an argument labelling, and to translate an argument labelling to a set of arguments, respectively) such that $\mathtt{Args2Lab}(\mathcal{A}rgs) = (\mathcal{A}rgs, \mathcal{A}rgs^+, Ar \setminus (\mathcal{A}rgs \cup \mathcal{A}rgs^+))$ and $\mathtt{Lab2Args}(\mathcal{L}ab) = \mathtt{in}(\mathcal{L}ab)$. It has been proven [10] that if $\mathcal{A}rgs$ is an admissible set (resp. a complete, grounded or preferred extension) then $\mathtt{Args2Lab}(\mathcal{A}rgs)$ is an admissible labelling (resp. a complete, grounded or preferred labelling), and that if $\mathcal{L}ab$ is an admissible labelling (resp. a complete, grounded or preferred labelling) then $\mathtt{Lab2Args}(\mathcal{L}ab)$ is an admissible set (resp. a complete, grounded or preferred extension). Moreover, when the domain and range of Args2Lab and Lab2Args are restricted to complete extensions and complete labellings they become injective functions and each other's reverses, which implies that the complete extensions (resp. the grounded extension and the preferred extensions) and the complete labellings (resp. the grounded labelling and the preferred labellings) are one-to-one related [10].

3. Strongly Admissible Sets

The concept of strong admissibility was first introduced by Baroni and Giacomin [2], using the notion of *strong defence*.

Definition 7 ([2]). *Let (Ar, att) be an argumentation framework, $A \subseteq Ar$ and $\mathcal{A}rgs \subseteq Ar$ be a set of arguments. A is* strongly defended *by $\mathcal{A}rgs$ iff each attacker B of A is attacked by some $C \in \mathcal{A}rgs \setminus \{A\}$ such that C is strongly defended by $\mathcal{A}rgs \setminus \{A\}$.*

Baroni and Giacomin say that a set $\mathcal{A}rgs$ satisfies the strong admissibility property iff it strongly defends each of its arguments [2]. However, it is also possible to define strong admissibility without having to refer to strong defence.

Definition 8. *Let (Ar, att) be an argumentation framework. $\mathcal{A}rgs \subseteq Ar$ is* strongly admissible *iff every $A \in \mathcal{A}rgs$ is defended by some $\mathcal{A}rgs' \subseteq \mathcal{A}rgs \setminus \{A\}$ which in its turn is again strongly admissible.*
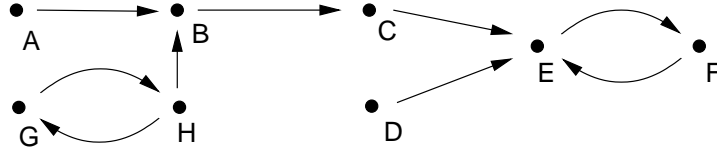
Figure 1. Example argumentation framework.

Theorem 1. *Let (Ar, att) be an argumentation framework and $\mathcal{A}rgs \subseteq Ar$. $\mathcal{A}rgs$ is a strongly admissible set (in the sense of Definition 8) iff each $A \in \mathcal{A}rgs$ is strongly defended by $\mathcal{A}rgs$ (in the sense of Definition 7).*

Proof. See [7]. □

Theorem 2. *Let (Ar, att) be an argumentation framework and let $\mathcal{A}rgs \subseteq Ar$ be a strongly admissible set. It holds that:*
- $\mathcal{A}rgs$ *is conflict-free*
- $\mathcal{A}rgs$ *is admissible*

Proof. Conflict-freeness follows from [2, Proposition 51], together with Theorem 1. Admissibility follows from conflict-freeness, together with the fact that every strongly admissible set defends each of its arguments. □

To illustrate the concept of strong admissibility, consider the argumentation framework of Figure 1. Here, the strongly admissible sets are $\emptyset$, $\{A\}$, $\{A, C\}$, $\{A, C, F\}$, $\{D\}$, $\{A, D\}$, $\{A, C, D\}$, $\{D, F\}$, $\{A, D, F\}$ and $\{A, C, D, F\}$, the latter also being the grounded extension. As an example, the set $\{A, C, F\}$ is strongly admissible as A is defended by $\emptyset$, C is defended by $\{A\}$ and F is defended by $\{A, C\}$, each of which is a strongly admissible subset of $\{A, C, F\}$ not containing the argument it defends.[4]

Baroni and Giacomin prove that the grounded extension is the unique biggest strongly admissible set [2].[5] However, it can additionally be proved that the strongly admissible sets form a lattice, of which the grounded extension is the top element and the empty set is the bottom element. To do so, we need two lemmas.

Lemma 1. *If $\mathcal{A}rgs_1$ and $\mathcal{A}rgs_2$ are strongly admissible sets, then $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ is also a strongly admissible set.*

Lemma 2. *Each admissible set has a unique biggest (w.r.t. set-inclusion) strongly admissible subset.*

If $\mathcal{A}rgs$ is an admissible set, we write $\mathcal{A}rgs{\Downarrow}$ for its biggest strongly admissible subset.

Theorem 3. *Let (Ar, att) be an argumentation framework. The strongly admissible sets of this framework form a lattice (w.r.t. set inclusion).*

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[4]Please notice that although the set $\{A, F\}$ defends argument C in $\{A, C, F\}$, it is in its turn not strongly admissible (unlike $\{A\}$). Hence the requirement in Definition 8 for $\mathcal{A}rgs'$ to be a *subset* of $\mathcal{A}rgs \setminus \{A\}$.

[5]Hence, each strongly admissible set is an admissible set that is contained in the grounded extension. The converse, however, does not hold. For instance, in Figure 1, $\{F\}$ is an admissible set that is contained in the grounded extension, but it is not a strongly admissible set.

Proof. We need to prove that each two strongly admissible sets have a supremum (a lowest upper bound) and a infimum (a greatest lower bound).

supremum Let $Args_1$ and $Args_2$ be two strongly admissible sets. From Lemma 1 it follows that $Args_1 \cup Args_2$ is again a strongly admissible set. Since, by definition, $Args_1 \subseteq Args_1 \cup Args_2$ and $Args_2 \subseteq Args_1 \cup Args_2$, it follows that $Args_1 \cup Args_2$ is an upper bound. Moreover, it is also a *lowest* upper bound, since any proper subset of $Args_1 \cup Args_2$ will not be a superset of $Args_1$ and $Args_2$.

infimum Let $Args_1$ and $Args_2$ be two strongly admissible sets. Let $Args_3$ be $Args_1 \cap Args_2$. From the fact that $Args_3$ is conflict-free, it follows that it has a (unique) biggest admissible subset, which we will refer to as $Args_3'$. From Lemma 2 it follows that $Args_3'$ has a (unique) biggest strongly admissible subset, which we will refer to as $Args_3''$. We now prove that $Args_3''$ is an infimum of $Args_1$ and $Args_2$.

lower bound From the fact that $Args_3'' \subseteq Args_3' \subseteq Args_3 = Args_1 \cap Args_2$ it follows that $Args_3'' \subseteq Args_1$ and $Args_3'' \subseteq Args_2$.

greatest lower bound Let $Args_3'''$ be a strongly admissible admissible set such that $Args_3''' \subseteq Args_1$ and $Args_3''' \subseteq Args_2$. Then, by definition, $Args_3''' \subseteq Args_3$. Since $Args_3'''$ is admissible, it follows that $Args_3''' \subseteq Args_3'$ (since $Args_3'$ is the *biggest* admissible subset of $Args_3$). Since $Args_3'''$ is a strongly admissible subset of $Args_3'$ it follows that $Args_3''' \subseteq Args_3''$ (since $Args_3''$ is the *biggest* strongly admissible subset of $Args_3'$). $\square$

In essence, if $Args_1$ and $Args_2$ are strongly admissible sets, then $Args_1 \cup Args_2$ is their supremum, and $(Args_1 \cap Args_2)\!\downarrow\!\Downarrow$ is their infimum. By forming a lattice, with the empty set as its bottom element and the grounded extension as its top element, the strongly admissible sets differ from the admissible sets, which form a semi-lattice with the empty set as its bottom element, and the preferred extensions as its top elements [15]. It also distinguishes the strongly admissible sets from the complete extensions, which form a semi-lattice with the grounded extension as its bottom element and the preferred extensions as its top elements [15].

4. Strongly Admissible Labellings

Argument labellings [5,10] have become a popular approach for purposes such as argumentation algorithms [6,16,18], argument-based judgment aggregation [11,12] and issues of measuring distance of opinion [4]. In the current section, we develop a labelling account of strong admissibility, which will subsequently be used to analyse some of the existing discussion games for grounded semantics.

To define a strongly admissible labelling, we first have to introduce the concept of a min-max numbering.

Definition 9. *Let $\mathcal{L}ab$ be an admissible labelling of argumentation framework (Ar, att). A min-max numbering is a total function $\mathcal{MM}_{\mathcal{L}ab} : \text{in}(\mathcal{L}ab) \cup \text{out}(\mathcal{L}ab) \to \mathbb{N} \cup \{\infty\}$ such that for each $A \in \text{in}(\mathcal{L}ab) \cup \text{out}(\mathcal{L}ab)$ it holds that:*

- *if $\mathcal{L}ab(A) = \text{in}$ then $\mathcal{MM}_{\mathcal{L}ab}(A) = max(\{\mathcal{MM}_{\mathcal{L}ab}(B) \mid B$ attacks A and $\mathcal{L}ab(B) = \text{out}\}) + 1$ (with $max(\emptyset)$ defined as 0)*

- *if $\mathcal{L}ab(A) = $ out then $\mathcal{MM}_{\mathcal{L}ab}(A) = min(\{\mathcal{MM}_{\mathcal{L}ab}(B) \mid B$ attacks A and $\mathcal{L}ab(B) = $ in$\}) + 1$ (with $min(\emptyset)$ defined as ∞)*

Theorem 4. *Every admissible labelling has a unique min-max numbering.*

Given an admissible labelling, its min-max numbering can be computed in an inductive, bottom-up way. Each step for doing so consists of two substeps. In the first substep, we identify the unnumbered in-labelled arguments of which all out-labelled attackers have already been numbered, and number them accordingly (with the maximal min-max number of their out-labelled attackers, plus 1). In the second substep, we identify the unnumbered out-labelled arguments that have at least one in-labelled attacker that has already been numbered, and number them accordingly (with the minimal min-max number of their in-labelled attackers that have already been numbered, plus 1).[6] We keep on doing such steps until no new arguments become numbered. Those in and out-labelled arguments that are still unnumbered then become numbered with ∞.

As an example, in the argumentation framework of Figure 1 consider the admissible labelling $(\{A, C, F, H\}, \{B, E, G\}, \{D\})$. In step 1 (first substep) the in-labelled argument A is numbered with 1, as it has no attackers. In the second substep, the out-labelled argument B is numbered with 2, as it has an in-labelled attacker that is already numbered (A). In step 2, the in-labelled argument C is numbered with 3, and the out-labelled argument E is numbered with 4. In step 3, the in-labelled argument F is numbered with 5. After that, no subsequent steps will yield any additional numbers, so the remaining unnumbered in and out-labelled arguments (G and H) are numbered ∞. Notice that because D is labelled undec, it will remain unnumbered.

It can be verified that the procedure sketched above yields a correct min-max numbering [7]. Moreover, it turns out that every min-max numbering of the same admissible labelling has to be equal to the one yielded by the above sketched procedure, thus obtaining uniqueness [7].[7]

Definition 10. *A strongly admissible labelling is an admissible labelling whose min-max numbering yields natural numbers only (so no argument is numbered ∞).*

From Definition 10 it directly follows that every strongly admissible labelling is also an admissible labelling.

Theorem 5. *Let (Ar, att) be an argumentation framework.*
- *for every strongly admissible set $\mathcal{A}rgs \subseteq Ar$, it holds that Args2Lab($\mathcal{A}rgs$) is a strongly admissible labelling*
- *for every strongly admissible labelling $\mathcal{L}ab$, it holds that Lab2Args($\mathcal{L}ab$) is a strongly admissible set*

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[6]It can be observed that if in subsequent steps, more of its in-labelled attackers become numbered, the additional min-max numbers of these in-labelled attackers will never be lower than the min-max number of the first in-labelled attacker that became numbered. Hence, the *minimal* min-max number of the in-labelled attackers will remain the same. This gives us the desirable property that once an argument becomes numbered, it never has to be renumbered later on.

[7]It has to be mentioned, however, that *different* admissible labellings yield different min-max numberings. For instance, the admissible labelling $(\{D, F\}, \{E\}, \{A, B, C, G, H\})$ numbers argument F with 3 instead of with 5.

Theorem 5 can be proven using induction on the steps of the above sketched numbering procedure. We refer to [7] for details.[8] Similar to the strongly admissible sets, the strongly admissible labellings form a lattice (w.r.t. $\sqsubseteq$) with the all-undec labelling as the bottom element and the grounded labelling as the top element. The proof of this follows a structure similar to that of Theorem 3.

5. Strong Admissibility and Argument Games

Now that some of the formal properties of strong admissibility have been examined, the next step is to study some of its applications. In particular, it turns out that strong admissibility is one of the corner stones of the discussion games for grounded semantics.

5.1. The Standard Grounded Game

We first describe the Standard Grounded Game [16,20].

Definition 11. *A* discussion *in the Standard Grounded Game is a finite sequence* $[A_1, \ldots, A_n]$ *($n \geq 1$) of arguments (sometimes called* moves*), of which the odd moves are called P-moves (Proponent moves) and the even moves are called O-moves (Opponent moves), such that:*

1. *every O-move is an attacker of the preceding P-move (that is, every A_i where i is even and $2 \leq i \leq n$ attacks A_{i-1})*
2. *every P-move except the first one is an attacker of the preceding O-move (that is, every A_i where i is odd and $3 \leq i \leq n$ attacks A_{i-1})*
3. *P-moves are not repeated (that is, for every odd $i, j \in \{1, \ldots, n\}$ it holds that if $i \neq j$ then $A_i \neq A_j$)*

A discussion is called terminated *iff there is no A_{n+1} such that $[A_1, \ldots, A_n, A_{n+1}]$ is a discussion. A terminated discussion is said to be* won *by the player making the last move.*

An *argument tree* is a tree of which each node (n) is labelled with an argument ($Arg(\mathbf{n})$). The *level* of a node is the number of nodes in the path to the root.

Definition 12. *A* winning strategy *of the Standard Grounded Game for argument A is an argument tree, where the root is labelled with A, such that*

1. *for each path from the root ($\mathbf{n}_{root}$) to a leaf node ($\mathbf{n}_{leaf}$) it holds that the arguments on this path form a terminated discussion won by P*
2. *for each node at odd level $\mathbf{n}_P$ it holds that $\{Arg(\mathbf{n}_{child}) \mid \mathbf{n}_{child} \text{ is a child of } \mathbf{n}_P\}$ $= \{B \mid B \text{ attacks } Arg(\mathbf{n}_P)\}$ and the number of children of $\mathbf{n}_P$ is equal to the number of attackers of $Arg(\mathbf{n}_P)$*
3. *each node of even level $\mathbf{n}_O$ has precisely one child $\mathbf{n}_{child}$, and $Arg(\mathbf{n}_{child})$ attacks $Arg(\mathbf{n}_O)$*

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[8] Notice that running the numbering procedure on an arbitrary admissible labelling identifies its unique maximal (w.r.t. $\sqsubseteq$) strongly admissible sublabelling, consisting of those in and out-labelled arguments that are assigned natural numbers, and all other arguments becoming undec. Hence, every admissible labelling has a unique maximal strongly admissible sublabelling. Using Theorem 5, we then obtain that also every admissible set has a unique biggest (w.r.t. $\subseteq$) strongly admissible subset, hence proving Lemma 2.

The correctness and completeness of the Standard Grounded Game depends on the presence of a winning strategy. That is, an argument A is in the grounded extension iff there exists a winning strategy for A. Interesting enough, it turns out that such a winning strategy defines a strongly admissible set containing A.

Theorem 6. *The set of all proponent moves in a winning strategy of the Standard Grounded Game is strongly admissible.*

Proof. We prove this by induction over the depth (i) of the winning strategy game tree.

basis $i = 0$. In that case, the winning strategy consists of a single argument (say, A). This means that A has no attackers. Hence, $\{A\}$ is a strongly admissible set.

step Suppose that every winning strategy of depth less or equal than i has its proponent moves constituting a strongly admissible set. We need to prove that also every winning strategy of depth $i + 2$ has its proponent moves constituting a strongly admissible set. Let WS be a winning strategy of depth $i + 2$. Let A be the argument at the root of the tree. Let $WS'_1, \ldots, WS'_n$ be the subtrees whose roots are at distance 2 of the root of WS. The induction hypothesis states that for each of these subtrees (WS'_j), their set of proponent moves $Args'_j$ constitutes a strongly admissible set. Therefore (by Lemma 1) the set $Args' = \cup_{j=1}^{n} Args'_j$ is strongly admissible. Also, $A \notin Args'$ (this is because the proponent is not allowed to repeat his moves). Let B be an arbitrary argument in $Args$ (the set of all proponent moves in the winning strategy). We distinguish two cases:

1. $B \in Args'$. Then, since $Args'$ is a strongly admissible set, there exists an $Args'' \subseteq Args' \setminus \{B\}$ that defends B and is itself strongly admissible. Since $Args' \subseteq Args$, it also holds that $Args'' \subseteq Args \setminus \{B\}$.

2. $B \notin Args'$. Then $B = A$ (the root of the tree WS). The structure of the WS tree is such that B is defended by the roots of $WS'_1, \ldots, WS'_n$. So B is defended by the strongly admissible set $Args'$. Also $B \notin Args'$, so $Args' \subseteq Args \setminus \{B\}$, therefore satisfying Definition 8. $\square$

It can also be observed that a winning strategy defines a strongly admissible labelling.

Theorem 7. *Let $Args_P$ be the set of proponent moves and $Args_O$ be the set of opponent moves of a particular winning strategy given an argumentation framework (Ar, att). It holds that $(Args_P, Args_O, Ar \setminus (Args_P \cup Args_O))$ is a strongly admissible labelling.*

Proof. Given that $Args_P$ is strongly admissible (Theorem 6) it then follows from Theorem 5 that $Lab_{PP+} = (Args_P, Args_P^+, Ar \setminus (Args_P \cup Args_P^+))$ is a strongly admissible labelling. Now consider $Lab_{PO} = (Args_P, Args_O, Ar \setminus (Args_P \cup Args_O))$. Notice that $Args_P^- \subseteq Args_P^+$, otherwise $Args_P$ would not be an admissible set. Also, from the structure of a winning strategy (with the Opponent playing all possible attackers of each Proponent move as its children) it follows that $Args_O = Args_P^-$. Hence, $Args_O \subseteq Args_P^+$. Lab_{PO} has the same min-max numbering as Lab_{PP+} (minus the arguments that are no longer out in Lab_{PO}, since $\mathrm{out}(Lab_{PO}) \subseteq \mathrm{out}(Lab_{PP+})$, as $Args_O \subseteq Args_P^+$). This is because the out-labelled arguments in $Args_P^+ \setminus Args_O$ do not influence the min-max numbers of the in-labelled arguments in $Args_P$. It then follows that the min-max numbers of the out-labelled arguments in Lab_{PO} also stay the same. Hence, the min-max numbering of Lab_{PO} is essentially a restricted version (with a smaller domain) of the

min-max numbering of $\mathcal{L}ab_{PP+}$. So from the fact that $\mathcal{L}ab_{PP+}$ is a strongly admissible labelling (not yielding ∞) it directly follows that $\mathcal{L}ab_{PO}$ is a strongly admissible labelling (not yielding ∞). □

Hence, given a winning strategy of the Standard Grounded Game, the set of all proponent moves and the set of all opponent moves essentially define a strongly admissible labelling.

5.2. The Grounded Persuasion Game

The second discussion game to be discussed is the Grounded Persuasion Game [13], which can be seen as a type of Mackenzie-style dialogue, applied to abstract argumentation. The game has two participants (proponent P and opponent O) and four types of moves: `claim` (the first move in the discussion, with which P utters the main claim that a particular argument has to be labelled `in`), `why` (with which O asks why a particular argument has to be labelled in a particular way), `because` (with which P explains why a particular argument has to be labelled a particular way) and `concede` (with which O indicates agreement with a particular statement of P). During the game, both P and O keep *commitment stores*, partial labellings (which we will refer to as $\mathcal{L}ab_P$ and $\mathcal{L}ab_O$) which keep track of which arguments they think are `in` and `out` during the course of the discussion. For P, a commitment is added every time he utters a `claim` or `because` statement. For O, a commitment is added every time he utters a `concede` statement. An *open issue* is an argument where only one player has a commitment. Since the game is such that at each stage, $\mathcal{L}ab_O \sqsubseteq \mathcal{L}ab_P$, this means an argument where P already has a commitment while O has not. Some of the key rules of the Grounded Persuasion Game are as follows.

- If O utters a `why in`(A) statement (resp. a `why out`(A) statement) then P has to reply with `because out`$(B_1, \ldots, B_n)$ where $B_1, \ldots, B_n$ are all attackers of A (resp. with `because in`(B) where B is an attacker of A).
- Any `why in`(A) or `why out`(A) statement of O has to be related to the most recently created open issue in the discussion.
- A `because` statement is not allowed to use an argument that is already an open issue.
- Every time an open issue is resolved, O has to concede immediately. That is, every time O has enough evidence to agree with P that a particular argument has to be labelled `in` (because for each of its attackers, O is already committed that the attacker is labelled `out`) or has to be labelled `out` (because it has an attacker of which O is already committed that it is labelled `in`) then O has to utter the relevant `concede` statement immediately.

We refer to [13] for full formal details of the game. An example discussion of the Grounded Persuasion Game can be found in Figure 2 (bottom).

Unlike the Standard Grounded Game, in the Grounded Persuasion Game it is not necessary to construct a winning strategy to show grounded membership. Instead, an argument A is in the grounded extension iff there exists *at least one game* that starts with P uttering "`claim in`(A)" and is won by P [13].[9]

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[9]A discussion is won by P iff at the end of the game O is committed that the argument the discussion started with is labelled `in`.

As a general property of the Grounded Persuasion Game, it can be observed that at every stage of the discussion, the commitment store of O ($\mathcal{L}ab_O$) forms an admissible labelling.[10] This is because whenever a new `in`-commitment is added, O is already committed that all its attackers are `out`, and whenever a new `out`-commitment is added, O is already committed that at least one attacker is `in`. Moreover, the commitment store of O also forms a *strongly* admissible labelling. This is because every time a new `in`-commitment is added, all its `out`-attackers have natural min-max numbers, and every time a new `out`-commitment is added, it has an `in`-attacker with a natural min-max number. Although it is possible for the `out`-commitments to obtain lower min-max numbers later on in the game (in case it gets new `in`-attackers) the fact that each commitment has a natural min-max number when it is first created implies that it will continue to have a natural min-max number at any further point of the game. Hence, we obtain the following result.

Theorem 8. *If, given an argumentation framework* (Ar, att)*, a particular discussion under the Grounded Persuasion Game is won by P, then the resulting commitment store of O ($\mathcal{L}ab_O$) forms the strongly admissible labelling* $(\mathtt{in}(\mathcal{L}ab_O), \mathtt{out}(\mathcal{L}ab_O), Ar \setminus (\mathtt{in}(\mathcal{L}ab_O) \cup \mathtt{out}(\mathcal{L}ab_O)))$*.*

5.3. The Standard Grounded Game (SGG) vs. the Grounded Persuasion Game (GPG)

So far, we have seen that both the SGG and the GPG show membership of the grounded extension essentially by building a strongly admissible labelling where the argument in question is labelled `in`.[11] This raises the question of how many steps each of these games requires for doing so. Consider again the argumentation framework of Figure 2 (top left). The winning strategy of the SGG is in the same figure (top right). Now consider what would happen if one would start to extend the argumentation framework by duplicating the middle part. That is, suppose we have arguments $B_1, \dots, B_n$ and $C_1, \dots, C_n$ (with n being an odd number), as well as arguments A and D. Suppose that for every $i \in \{1, \dots, n-1\}$ B_{i+1} attacks B_i, and C_{i+1} attacks C_i, and that for each *even* $i \in \{2, \dots n-1\}$ B_{i+1} attacks C_i, and C_{i+1} attacks B_i, and that B_1 and C_1 attack A, and that D attacks B_n and C_n. In that case, the branches in the SGG winning strategy would split at every O-move. So for $n = 3$ (as is the case in Figure 2) the number of branches is four, for $n = 5$ it is eight, etc. In general, the number of branches in the SGG winning strategy is $2^{(n+1)/2}$, with the number of nodes in the SGG winning strategy being $1 + 2\Sigma_{i=1}^{(n+1)/2} 2^i$. Hence, the number of steps needed in a winning strategy of the SGG can be *exponential* in relation to the size (number of `in` and `out` labelled arguments) of the strongly admissible labelling that the SGG winning strategy is constructing.[12]

As for the GPG, the situation is different. We observe that, as a general property, the total number of moves in a successful GPG (won by P) is at most three times the size of the strongly admissible labelling. This is because every `in` or `out`-labelled argument will have at most one associated `why` statement and precisely one associated `concede` statement, and the total number of `claim` and `because` statements will be less or equal

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[10]That is, if one regards all arguments where O does not have any commitments to be labelled `undec`.

[11]Similarly, it can be observed that for instance the credulous preferred game [9,21] shows membership of a preferred extension essentially by building an admissible labelling around the argument in question.

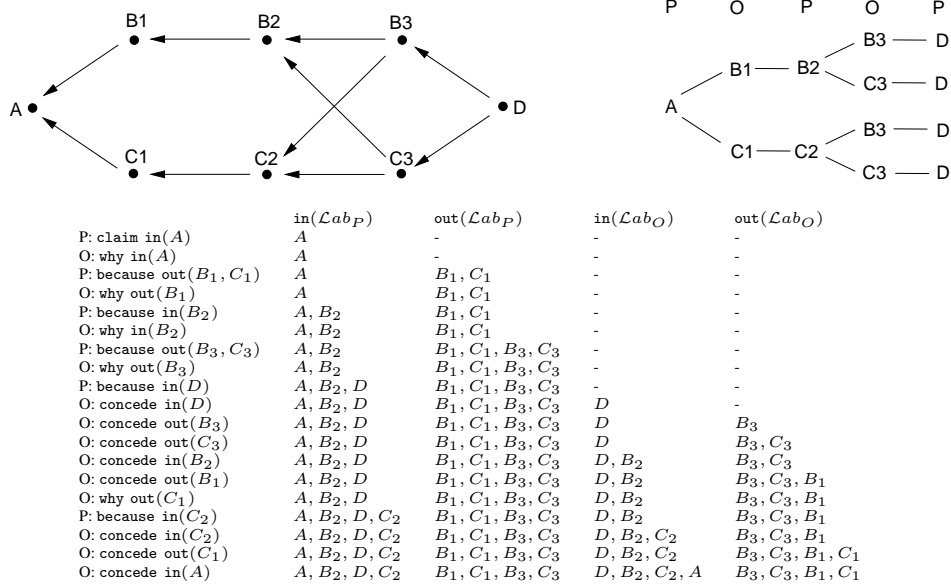[12]We thank Mikołaj Podlaszewski for this example.

Figure 2. The Standard Grounded Game (SGG) versus the Grounded Persuasion Game (GPG).

	in($\mathcal{L}ab_P$)	out($\mathcal{L}ab_P$)	in($\mathcal{L}ab_O$)	out($\mathcal{L}ab_O$)
P: claim in(A)	A	-	-	-
O: why in(A)	A	-	-	-
P: because out(B_1, C_1)	A	B_1, C_1	-	-
O: why out(B_1)	A	B_1, C_1	-	-
P: because in(B_2)	A, B_2	B_1, C_1	-	-
O: why in(B_2)	A, B_2	B_1, C_1	-	-
P: because out(B_3, C_3)	A, B_2	B_1, C_1, B_3, C_3	-	-
O: why out(B_3)	A, B_2	B_1, C_1, B_3, C_3	-	-
P: because in(D)	A, B_2, D	B_1, C_1, B_3, C_3	-	-
O: concede in(D)	A, B_2, D	B_1, C_1, B_3, C_3	D	-
O: concede out(B_3)	A, B_2, D	B_1, C_1, B_3, C_3	D	B_3
O: concede out(C_3)	A, B_2, D	B_1, C_1, B_3, C_3	D	B_3, C_3
O: concede in(B_2)	A, B_2, D	B_1, C_1, B_3, C_3	D, B_2	B_3, C_3
O: concede out(B_1)	A, B_2, D	B_1, C_1, B_3, C_3	D, B_2	B_3, C_3, B_1
O: why out(C_1)	A, B_2, D	B_1, C_1, B_3, C_3	D, B_2	B_3, C_3, B_1
P: because in(C_2)	A, B_2, D, C_2	B_1, C_1, B_3, C_3	D, B_2	B_3, C_3, B_1
O: concede in(C_2)	A, B_2, D, C_2	B_1, C_1, B_3, C_3	D, B_2, C_2	B_3, C_3, B_1
O: concede out(C_1)	A, B_2, D, C_2	B_1, C_1, B_3, C_3	D, B_2, C_2	B_3, C_3, B_1, C_1
O: concede in(A)	A, B_2, D, C_2	B_1, C_1, B_3, C_3	D, B_2, C_2, A	B_3, C_3, B_1, C_1

to the total number of concede statements. Hence, the total number of moves in the GPG is *linear* in relation to the size of the strongly admissible labelling that the GPG is constructing.

6. Discussion and Future Research

In the current paper, we have re-examined the concept of strong admissibility, from both theoretical and practical perspectives. From theoretical perspective, we have observed that the strongly admissible sets form a lattice with the empty set as bottom element and the grounded extension as top element. Also, we have developed the concept of a strongly admissible labelling, and shown how it relates to the concept of a strongly admissible set. From practical perspective, we have examined how strongly admissible labellings lie at the basis of both the Standard Grounded Game [16] and the Grounded Persuasion Game [13]. Although both essentially construct a strongly admissible labelling around the argument in question, the Grounded Persuasion Game does so using a linear number of steps, whereas the Standard Grounded Game can require an exponential number of steps.

One of the things we plan to examine in the near future is how the concept of strong admissibility can be useful in identifying the shortest discussion that shows an argument (A) is in the grounded extension. For instance, we conjecture that for each minimal (w.r.t. $\sqsubseteq$) strongly admissible labelling that labels A in, there exists a discussion under the Grounded Persuasion Game for argument A that builds precisely this labelling. However, there can be more than one such labelling. For argument F in Figure 1, for instance, both $(\{A, C, F\}, \{B, E\}, \{D, G, H\})$ and $(\{D, F\}, \{E\}, \{A, B, C, G, H\})$ are minimal (w.r.t. $\sqsubseteq$) strongly admissible labellings that label F in, but the size of the sec-

ond labelling is smaller than that of the first labelling, thus yielding a shorter discussion. How to precisely obtain such a strongly admissible labelling with minimal size is a topic for further investigation.

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